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THE MATHEMATICAL ANALYSIS OF A SIMPLE DUEL

By Arthur D. Groves

Stochastic duels

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BALLISTIC RESEARCH LABORATORIES
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ABSTRACT

The principles and techniques of simple Markov processes are used to analyze a simple duel to determine the limiting state probabilities (i.e., the probabilities of occurrence of the various possible outcomes of the duel). The duel is one in which A fires at B at a rate of r_A shots per minute starting at time $t = 0$ and with a single shot kill probability of P_A, and B fires at A at a rate of r_B shots per minute starting at time $t = \delta$ and with a single shot kill probability of P_B, and this exchange of fire continues until either A or B, or both are killed.

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THE MATHEMATICAL ANALYSIS OF A SIMPLE DUEL

The situation to be considered is one in which two players engage in a duel where they exchange fire, each at his own rate of fire. Let A denote the player who fires first, and let the other player, B, react to A's initial firing with a delay time of δ seconds. Thus A fires his first shot at time $t = 0$ seconds and B fires his first shot at time $t = \delta$ seconds. After their initial shots, A and B fire continuously at rates of fire of r_A shots per minute and r_B shots per minute, respectively. Thus the time at which A fires his p^{th} shot (denoted $t_{p,A}$) is given by

$$(1) \qquad t_{p,A} = \frac{60}{r_A}(p-1) \quad \text{seconds,}$$

and the time $t_{q,B}$ at which B fires his q^{th} shot is given by

$$(2) \qquad t_{q,B} = \delta + \frac{60}{r_B}(q-1) \quad \text{seconds.}$$

It will generally be impossible to predict with certainty the outcome of such a duel. The best that can be done is to determine the probabilities of the various possible outcomes of the duel. Beyond that, it may be of interest to know the probabilities that each of the possible outcomes has occurred after a specified number of shots have been exchanged. The computational model to be used to determine these probabilities will be in the language of Markov* processes.

The duel can be said to be in one of four <u>states</u> at any time. These states are

1. A and B are both alive (neither has won the duel)
2. B is dead; A is alive (A has won the duel)
3. A is dead; B is alive (B has won the duel)
4. A and B are both dead.

A player must kill his opponent while remaining alive himself to win the duel. The 4^{th} state may result when A and B exchange shots simultaneously. Any time a shot is fired by either A or B, a <u>transition</u> from one state to another takes place. (This transition may be from one state into the same state.) Thus the dueling process may be indexed by a discrete variable (such as the number of shots fired) rather than the continuous variable, time. Let

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* Howard, "Dynamics Programming and Markov Processes" The MIT Press, 1960.

(3) $$S_n = (s_{1n}, s_{2n}, s_{3n}, s_{4n})$$

be a 4-dimensional vector whose i^{th} component s_{in} is the probability that the duel is in the i^{th} state after n transitions. Define three transition matrices

(4) $$T_A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{Vmatrix}$$

$$T_B = \begin{Vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{Vmatrix}$$

$$T_C = \begin{Vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{Vmatrix}$$

where a_{ij} is the probability that the duel undergoes transition from state i to state j when A alone fires a shot, b_{ij} is the probability that the duel undergoes transition from state i to state j when B alone fires, and c_{ij} is the probability that the duel undergoes transition from state i to state j when A and B fire simultaneous shots.

Let P_A be the probability that A kills B on any given shot (i.e., the single shot kill probability for A when firing at B), and let P_B be the single shot kill probability for B when firing at A. Then the transition matrices can be rewritten

(5) $$T_A = \begin{Vmatrix} 1-P_A & P_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_B = \begin{Vmatrix} 1-P_B & 0 & P_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_C = \begin{Vmatrix} (1-P_A)(1-P_B) & P_A(1-P_B) & P_B(1-P_A) & P_AP_B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

Since the duel always starts in state 1 (with both A and B alive), the initial state vector S_o will be

(6) $$S_o = (1, 0, 0, 0).$$

Then S_n, the state vector after n transitions, is given by

(7) $$S_n = S_0 T_1 T_2 T_3 \ldots\ldots T_n,$$

where each T_Ω, $(\Omega = 1, 2, \ldots\ldots n)$ is either T_A, T_B, or T_C, depending on the order in which A and B fire during the process of dueling.

To carry this analysis further, it is necessary to know the order in which the matrices T_A, T_B, and T_C appear in equation 7. A has been designated as the initial firer, but, after that, the order depends on δ, r_A, and r_B. Since B fires his first shot at time $t = \delta$, the number of shots A can fire prior to that (denoted k) is determined by the relation

(8) $$k = [x]$$

where the bracket denotes the greatest integer less than x, and x is determined by equations 1 and 2 by setting

(9) $$t_{x,A} = t_{1,B}.$$

This can be rewritten

(10) $$x = \frac{\delta r_A}{60} + 1 \quad .$$

Thus

(11) $$k = \left[\frac{\delta r_A}{60} + 1 \right] ,$$

and the first k shots of the duel are fired by A. Therefore equation 7 becomes

(12) $$S_n = \begin{cases} S_o \, T_A{}^n & \text{for } n \le k \\ S_o \, T_A{}^k \; T_{k+1} \, T_{k+2} \cdots\cdots T_n & \text{for } n > k \end{cases}$$

Starting with T_{k+1}, the sequence of matrices in equation 12 will be periodic (for large n). This can be shown as follows. B fires his 1^{st} shot at $t = \delta$, and A fires his $(k+1)^{st}$ shot at $t \geq \delta$ seconds. The time interval between B's 1^{st} shot and A's $(k+1)^{st}$ shot is

(13) $$t_{k+1,A} - t_{1,B} = \frac{60k}{r_A} - \delta \quad \text{seconds.}$$

If it can be shown that this same time interval occurs between the shot fired by B at time $t_{1,B} + \mu\lambda$ and the shot fired by A at time $t_{k+1,A} + \mu\lambda$, ($\mu = 1, 2, \ldots$), then the sequence will be periodic with period λ seconds. Let the rates of fire, which are assumed to be rational numbers, be expressed as follows:

$r_A = \frac{a}{b}$ and $r_B = \frac{c}{d}$ shots per minute, where a, b, c, and d are all integers. Let

(14) $$z = \text{L.C.M } (b,d),$$

i.e., the least common multiple of b and d. Then $\frac{z}{b}$ and $\frac{z}{d}$ are both integers, and hence

$$z r_A = a\left(\frac{z}{b}\right)$$

and $\qquad zr_B = c\left(\frac{z}{d}\right)$ are also integers.

Let $\Delta = \text{L.C.M.}\ (zr_A, zr_B)$, and $\lambda = \frac{60\Delta}{zr_A r_B}$ seconds.

Now in λ seconds A will fire $\lambda\left(\frac{r_A}{60}\right)$ shots and B will fire $\lambda\left(\frac{r_B}{60}\right)$ shots. But

$$\lambda\left(\frac{r_A}{60}\right) = \frac{\Delta}{zr_B} \quad \text{and} \quad \lambda\left(\frac{r_B}{60}\right) = \frac{\Delta}{zr_A}$$

These are both integers since Δ is the least common multiple of zr_A and zr_B. Now let M_μ be the number of the shot that A would fire at time $\mu\lambda$ seconds after his $(k+1)^{st}$ shot, and let N_μ be the number of the shot which B would fire at time $\mu\lambda$ seconds after his 1^{st} shot. Then

$$(15) \qquad M_\mu = k+1+\mu\lambda\left(\frac{r_A}{60}\right)$$

and
$$N_\mu = 1+\mu\lambda\left(\frac{r_B}{60}\right) \quad , \quad \mu=1,\ 2,\ \ldots\ldots\ .$$

The times at which these firings take place are given by equations 1 and 2

$$t_{M_\mu,A} = \frac{60}{r_A}\left(k + \mu\lambda\left(\frac{r_A}{60}\right)\right) = \frac{60k}{r_A} + \mu\lambda$$

$$t_{N_\mu,B} = \delta + \frac{60}{r_B}\left(\mu\lambda\left(\frac{r_B}{60}\right)\right) = \delta + \mu\lambda$$

The difference between these two times is

$$t_{M_\mu,A} - t_{N_\mu,B} = \frac{60k}{r_A} - \delta = t_{k+1,A} - t_{1,B}\ .$$

Therefore the sequence is periodic with period

(16) $$\lambda = \frac{60\Delta}{zr_A r_B} \quad \text{seconds}$$

where $\Delta = \text{L.C.M.}\ (zr_A,\ zr_B)$

During each period A will fire $\lambda\left(\frac{r_A}{60}\right)$ shots and B will fire $\lambda\left(\frac{r_B}{60}\right)$ shots. That is,

$$\text{A will fire } \lambda\left(\frac{r_A}{60}\right) = \frac{\Delta}{zr_B} \text{ shots, and}$$

$$\text{B will fire } \lambda\left(\frac{r_B}{60}\right) = \frac{\Delta}{zr_A} \text{ shots.}$$

The total number of shots, M, fired in a cycle will be given by

(17) $$M = \frac{\Delta}{z}\left(\frac{1}{r_A} + \frac{1}{r_B}\right)$$

This establishes the maximum number of matrices in each cycle. If there are no simultaneous firings, the number of matrices will be M; $\frac{\Delta}{zr_B}$ of the matrices T_A and $\frac{\Delta}{zr_A}$ of the matrices T_B. If there is a simultaneous firing in the cycle, there will be $\left(\frac{\Delta}{zr_B} - 1\right)$ of the matrices T_A, $\left(\frac{\Delta}{zr_A} - 1\right)$ of the matrices T_B, and one of the matrices T_C in the cycle, for a total of (M-1) matrices.

Equation 12 can now be rewritten to express the state vector after m transition cycles (instead of after n transitions).

(18) $$S_{mM+k} = S'_m = S_o T_A{}^k\ (T_{k+1}\ T_{k+2} \cdots\cdots T_{k+M})^m$$

Because matrix multiplication is generally not commutative, it is not sufficient to know only how many of the M (or M-1) matrices in the cycle are of each type (T_A, T_B or T_C), but it must be determined whether $T_{k+\beta}$ is T_A, T_B or T_C for each β. In order to determine this firing order, the following procedure is probably the most simple: During the first of the m cycles, A fires his $(k+1)^{st}$, $(k+2)^{nd}$, ... $\left(k+\frac{\Delta}{zr_B}\right)^{th}$ shots, and B fires his 1^{st}, 2^{nd}, ... $\left(\frac{\Delta}{zr_A}\right)^{th}$ shots. The

time at which each of these firings takes place can be determined from equations 1 and 2, and then the firings in the cycle can be ordered by these times. After this has been done, let T denote the product of the M (or M-1) matrices, i.e.,

$$(19) \qquad T = T_{k+1}\, T_{k+2} \cdots T_{k+M} \ (\text{or } T_{k+M-1}) \ .$$

Define a new vector V as follows:

$$V = S_o T_A{}^k$$

It is easily shown by mathematical induction that if

$$T_A = \begin{Vmatrix} 1-P_A & P_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} ,$$

then

$$T_A{}^k = \begin{Vmatrix} (1-P_A)^k & P_A \sum_{i=0}^{k-1} (1-P_A)^i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} .$$

Thus

$$(20) \qquad V = S_o T_A{}^k = \left((1-P_A)^k,\ P_A \sum_{i=0}^{k-1} (1-P_A)^i,\ 0,\ 0 \right)$$

or $$V = (v_1,\ v_2,\ 0,\ 0) \quad .$$

where $$v_1 = (1-P_A)^k$$

and $$v_2 = P_A \sum_{i=0}^{k-1} (1-P_A)^i$$

In this new notation, equation 18 becomes

(21) $$S'_m = VT^m = (s'_{1m}, s'_{2m}, s'_{3m}, s'_{4m})$$

It is of interest to determine expressions for the s'_{im} , $(i = 1, 2, 3, 4)$ and to determine

(22) $$S = (s_1, s_2, s_3, s_4) = \lim_{m\to\infty} S'_m \quad .$$

which is the limit of the vector S'_m as $m\to\infty$. The i^{th} component of S is the probability that the duel will terminate in the i^{th} state, and the i^{th} component of S'_m is the probability that the duel will be in the i^{th} state after m transition cycles.

First note that the matrix T is of the form

(23) $$T = \begin{Vmatrix} t_1 & t_2 & t_3 & t_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

since it is the product of matrices of this form. The vector S'_m will be rewritten as the sum of two vectors, one of which is independent of m, and the other of which has components which approach zero as m increases. The z-transform analysis as described in Reference 1 will be used. Briefly, the z-transform can be described as follows:

For a sequence $\{f_n\}$ for which the sum $\sum_{n=0}^{\infty} f_n z^n$ is finite for some value or range of values of z, this sum is called the z-transform of f_n and is denoted F(z). Among its properties are the following five which will be applied to the analysis of the duel.

Property 1. z-Transform of a constant sequence.

If $f_n = c$ for $n = 0, 1, 2, \ldots\ldots$, then

$$F(z) = \sum_{n=0}^{\infty} cz^n = c \sum_{n=0}^{\infty} z^n = \frac{c}{1-z}$$

Property 2. z-Transform of a constant multiple of a sequence. If $g_n = cf_n$ for n = 0, 1, 2, ..., and F(z) and G(z) are the z-transforms of f_n and g_n respectively, then

$$G(z) = \sum_{n=0}^{\infty} cf_n z^n = c \sum_{n=0}^{\infty} f_n z^n = cF(z)$$

Property 3. z-transform of a geometric sequence. If $f_n = a^n$ for n = 0, 1, 2, ... , then

$$F(z) = \sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az}$$

Property 4. Recurrence relation of z-transforms. If F(z) is the z-transform of f_n and G(z) is the z-transform of f_{n+1}, then

$$G(z) = \sum_{n=0}^{\infty} f_{n+1} z^n = \sum_{j=1}^{\infty} f_j z^{j-1} = z^{-1} \sum_{j=1}^{\infty} f_j z^j$$

$$= z^{-1}\left(\sum_{j=0}^{\infty} f_j z^j - f_o \right) = z^{-1}\left(F(z) - f_o \right)$$

Property 5. z-transform of a sum. If $f_n = g_n + h_n$, for n = 0, 1, 2, ... , and F(z), G(z) and H(z) are the z-transforms of f_n, g_n, h_n respectively, then

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} (g_n + h_n) z^n$$

$$= \sum_{n=0}^{\infty} g_n z^n + \sum_{n=0}^{\infty} h_n z^n = G(z) + H(z)$$

The sequences of state probabilities s'_{im} in equation 21 (i = 1, 2, 3, 4; m = 0, 1, 2, ... ∞) satisfy the requirement for being amenable to z-transform analysis since they converge to limiting probabilities. The z-transform of a matrix or vector will simply mean the z-transform of each component of the matrix or vector.

Equation 21 can be rewritten

(24) $\quad S'_{m+1} = S'_m \, T$ where $m = 0, 1, \ldots,$ and $S'_o = V$

for equation 21 says

$$
\begin{aligned}
S'_1 &= VT = S'_0 \, T \\
S'_2 &= VT^2 = S'_1 \, T \\
S_3 &= VT^3 = S'_2 \, T \\
&\;\;\vdots \\
&\text{etc.}
\end{aligned}
$$

Taking z-transforms of both members of equation 24, the left hand member is transformed to

$z^{-1}\left(F(z) - V\right)$ by property 4 of the z-transform, where $F(z)$ is the z-transform of S'_m. The right hand member of equation 24 is transformed to $F(z)T$ by property 2 of the z-transform. Thus

(25) $$z^{-1}\left(F(z) - V\right) = F(z)T \; .$$

Solving for $F(z)$,

(26) $$F(z) = V(I - zT)^{-1}$$

where

$$I = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$ the multiplicative identity matrix.

The matrix $(I-zT)$ is given by

(27) $$(I-zT) = \begin{Vmatrix} 1-zt_1 & -zt_2 & -zt_3 & -zt_4 \\ 0 & 1-z & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1-z \end{Vmatrix}$$

using T as given in equation 23. The inverse of $(I-zT)$ is given by

$$(28)\qquad (I-zT)^{-1} = \begin{Vmatrix} \frac{1}{1-zt_1} & \frac{zt_2}{(1-z)(1-zt_1)} & \frac{zt_3}{(1-z)(1-zt_1)} & \frac{zt_4}{(1-z)(1-zt_1)} \\ 0 & \frac{1}{1-z} & 0 & 0 \\ 0 & 0 & \frac{1}{1-z} & 0 \\ 0 & 0 & 0 & \frac{1}{1-z} \end{Vmatrix}$$

Each element of $(I-zT)^{-1}$ can be written in the form

$\frac{\alpha}{1-z} + \frac{\beta}{1-zt_1}$ for some α and β and resulting matrix written as the sum of two matrices, as follows:

$$(29)(I-zT)^{-1} = \frac{1}{1-z}\begin{Vmatrix} 0 & \frac{t_2}{1-t_1} & \frac{t_3}{1-t_1} & \frac{t_4}{1-t_1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} + \frac{1}{1-t_1 z}\begin{Vmatrix} 1 & \frac{-t_2}{1-t_1} & \frac{-t_3}{1-t_1} & \frac{-t_4}{1-t_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

Substituting into equation 26,

$$(30)F(z) = V\left[\frac{1}{1-z}\begin{Vmatrix} 0 & \frac{t_2}{1-t_1} & \frac{t_3}{1-t_1} & \frac{t_4}{1-t_1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} + \frac{1}{1-t_1 z}\begin{Vmatrix} 1 & \frac{-t_2}{1-t_1} & \frac{-t_3}{1-t_1} & \frac{-t_4}{1-t_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}\right]$$

or

$$(31)F(z) = \frac{1}{1-z}\left(0,\ \frac{v_1 t_2}{1-t_1} + v_2,\ \frac{v_1 t_3}{1-t_1},\ \frac{v_1 t_4}{1-t_1}\right) + \frac{1}{1-t_1 z}\left(v_1,\ \frac{-v_1 t_2}{1-t_1},\ \frac{-v_1 t_3}{1-t_1},\ \frac{-v_1 t_4}{1-t_1}\right)$$

Now, taking the inverse z-transforms,

$$(32)S'_m = \left(0,\ \frac{v_1 t_2}{1-t_1} + v_2,\ \frac{v_1 t_3}{1-t_1},\ \frac{v_1 t_4}{1-t_1}\right) + t_1^m\left(v_1,\ \frac{-v_1 t_2}{1-t_1},\ \frac{-v_1 t_3}{1-t_1},\ \frac{-v_1 t_4}{1-t_1}\right)$$

Thus the probabilities that the duel is in each of the four states after m transition cycles and at the end of the duel are as follows:

Table 1. m-Cycle and Limiting Probabilities

State	Description	Probability: After m Cycles	Probability: At End of Duel
1	Neither A nor B wins	$v_1 t_1^{\,m}$	0
2	A wins	$v_2 + \frac{v_1 t_2}{1-t_1}(1-t_1^m)$	$\frac{v_1 t_2}{1-t_1} + v_2$
3	B wins	$\frac{v_1 t_3}{1-t_1}(1-t_1^m)$	$\frac{v_1 t_3}{1-t_1}$
4	A & B Kill Each Other	$\frac{v_1 t_4}{1-t_1}(1-t_1^m)$	$\frac{v_1 t_4}{1-t_1}$

The usefulness of this method depends on how easy it is to determine vector V and matrix T from the initial duel conditions. If A fires many rounds before B fires his first, then the determination of V may become tedious. If the period λ of the sequence of firers is long, it may consist of many transition matrices and the computation of T may be difficult. Indeed, in the unusual case where either r_A or r_B is irrational, the sequence of firers is not periodic, and the method fails. However, in most cases of interest, both V and T will be fairly easy to compute, and therefore the method will be useful.

Three sample duels will be used to illustrate the method.

In the first, let $P_A = .3$, $P_B = .4$, $\delta = 15$ seconds, and $r_A = r_B = 2$ shots per minute. Then z = L.C.M. (1,1) = 1. Also

$$\Delta = \text{L.C.M.}\ (zr_A,\ zr_B) = \text{L.C.M.}\ (2,\ 2) = 2$$

$$\lambda = \frac{60\Delta}{zr_A r_B} = 30 \text{ seconds.}$$

$$\frac{\Delta}{zr_B} = 1 \text{ shot per cycle by A}$$

$$\frac{\Delta}{zr_A} = 1 \text{ shot per cycle by B}$$

$$T_A = \begin{Vmatrix} .7 & .3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_B = \begin{Vmatrix} .6 & 0 & .4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_C = \begin{Vmatrix} .42 & .18 & .28 & .12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$k = \left[\frac{\delta r_A}{60} + 1\right] = 1$$

Therefore A fires one shot prior to the beginning of the first cycle, and the cycle consists of 2 shots - one by B followed by one by A. Thus

$$T = T_B T_A = \begin{Vmatrix} .6 & 0 & .4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} .7 & .3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} = \begin{Vmatrix} .42 & .18 & .40 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

From equation 20

$$v_1 = (1-P_A)^k = (.7)^1 = .7$$

$$v_2 = P_A \sum_{i=0}^{k-1} (1-P_A)^i = .3$$

and $t_1 = .42$

$t_2 = .18$

$t_3 = .40$

$t_4 = 0$

The following table, in the form of and using the results in Table 1, gives the results of this duel.

State	Description	Probability After m Cycles	Probability At End of Duel
1	Neither A nor B wins	$(.7)(.42)^m$	0
2	A wins	$.3 + .21724\ (1-.42^m)$	.51724
3	B wins	$.48276\ (1-.42^m)$	.48276
4	A & B Kill Each Other	0	0

The limiting probabilities from the above table can be verified by the following simple analysis which applies in this example where the duelers simply alternate fire, with A firing first.

Let $P_{n,A}$ be the probability that A wins the duel on his n^{th} shot. Then

$$P_{1,A} = P_A$$

$$P_{2,A} = (1-P_A)\ (1-P_B)\ P_A$$

$$P_{3,A} = (1-P_A)\ (1-P_B)\ (1-P_A)\ (1-P_B)\ P_A = (1-P_A)^2\ (1-P_B)^2\ P_A$$

$$\vdots$$

$$P_{n,A} = (1-P_A)^{n-1}\ (1-P_B)^{n-1}\ P_A.$$

Then, if π_A denotes the probability that A wins the duel,

$$\pi_A = \sum_{n=1}^{\infty} P_{n,A}$$

$$= P_A \sum_{n=1}^{\infty} (1-P_A)^{n-1}\ (1-P_B)^{n-1}$$

$$= \frac{P_A}{1-(1-P_A)(1-P_B)} \quad .$$

In the example, $\pi_A = \frac{.3}{1-(.7)(.6)} = .51724$

Similarly let $P_{m,B}$ be the probability that B wins the duel on his m^{th} shot. Then

$$P_{1,B} = (1-P_A)\ P_B$$

$$P_{2,B} = (1-P_A)(1-P_B)(1-P_A)\ P_B = (1-P_A)^2\ (1-P_B)\ P_B$$

$$P_{3,B} = (1-P_A)(1-P_B)(1-P_A)(1-P_B)(1-P_A)\ P_B = (1-P_A)^3\ (1-P_B)^2\ P_B$$

$$\vdots$$

$$P_{m,B} = (1-P_A)^m \; (1-P_B)^{m-1} \; P_B \quad .$$

Now the probability that B wins, denoted π_B, is given by

$$\pi_B = \sum_{m=1}^{\infty} P_{m,B}$$

$$= P_B \; (1-P_A) \sum_{m=1}^{\infty} (1-P_A)^{m-1} \; (1-P_B)^{m-1}$$

$$= \frac{P_B \; (1-P_A)}{1-(1-P_A)(1-P_B)}$$

In the example $\pi_B = \dfrac{(.4)(.7)}{1-(.7)(.6)} = .48276$

Since $\pi_A + \pi_B = \dfrac{P_A}{1-(1-P_A)(1-P_B)} + \dfrac{P_B(1-P_A)}{1-(1-P_A)(1-P_B)}$

$$= \frac{P_A + P_B(1-P_A)}{1-(1-P_A)(1-P_B)}$$

$$= \quad 1 \quad ,$$

the duel cannot possibly terminate in either state 1 or state 4.

As a second example consider the following somewhat more complex duel situation which does not lend itself to the simple analysis used as a check in the first example.

$P_A = .2 \qquad P_B = .8 \qquad \delta = 30$ seconds

$r_A = 5$ shots per minute $\qquad r_B = 2$ shots per minute

Here

$$T_A = \begin{Vmatrix} .8 & .2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_B = \begin{Vmatrix} .2 & 0 & .8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$T_C = \begin{Vmatrix} .16 & .04 & .64 & .16 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$k = \left[\frac{\delta r_A}{60} + 1\right] = \left[\frac{(30)(5)}{60} + 1\right] = 3$$

$$z = \text{L.C.M. } (1,1) = 1$$

$$\Delta = \text{L.C.M. } (zr_A, zr_B) = \text{L.C.M. } (5,2) = 10$$

$$\lambda = \frac{60\Delta}{zr_A r_B} = \frac{600}{10} = 60 \text{ seconds.}$$

A will fire $\frac{\Delta}{zr_B} = \frac{10}{2} = 5$ shots

B will fire $\frac{\Delta}{zr_A} = \frac{10}{5} = 2$ shots

Thus the period is 60 seconds long. During that time A will fire 5 shots and B will fire 2 shots. In particular, B will fire his 1^{st} and 2^{nd} shots, while A fires his 4^{th}, 5^{th}, 6^{th}, 7^{th} and 8^{th} shots. The following table gives the times at which each of these firings takes place; (using equations 1 and 2).

Shot	Times to Fire	
Number	A	B
1		30
2		60
3		
4	36	
5	48	
6	60	
7	72	
8	84	

Thus the firing order for the cycle is

B (at 30 seconds)
A (at 36 seconds)
A (at 48 seconds)
A & B simultaneously (at 60 seconds)
A (at 72 seconds)
A (at 84 seconds)

so

$$T = T_B \, T_A \, T_A \, T_C \, T_A \, T_A \quad .$$

Carrying out this multiplication of matrices,

$$T = \begin{Vmatrix} .0131072 & .0844928 & .8819200 & .0204800 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

From equation 20,

$$v_1 = .8^3 = .512$$
$$v_2 = .2\,(1 + .8 + .64) = .488$$

and from the matrix T above

$$t_1 = .0131072$$
$$t_2 = .0844928$$
$$t_3 = .8819200$$
$$t_4 = .0204800$$

The following table gives the probabilities which describe the state of the duel after m cycles and at the end of the duel.

State	Description	Probabilities: After m Cycles	Probabilities: At End of Duel
1	Neither A nor B wins	$(.512)(.0131072)^m$	0
2	A wins	$.488 + .043835(1-.0131072^m)$	.53183
3	B wins	$.45754(1-.0131072^m)$	.45754
4	A & B kill each other	$.01063(1-.0131072^m)$	.01063

In the first two examples the rates of fire have been integers so that $z=1$. In the last example this will not be the case. Let

$$r_A = \frac{1}{2} \text{ and } r_B = \frac{2}{3} \text{ shots per minute}$$

and $\qquad \delta = 10$ seconds and $P_A = .2$ and $P_B = .1$

Then $\qquad z = \text{L.C.M.}\ (2,3) = 6,$

$$\Delta = \text{L.C.M.}\ (zr_A,\ zr_B) = \text{L.C.M.}\ (3,4) = 12$$

$$\lambda = \frac{60\Delta}{zr_Ar_B} = 360 \text{ seconds per cycle.}$$

In each cycle A fires $\frac{\Delta}{zr_B} = \frac{12}{4} = 3$ shots and B fires $\frac{\Delta}{zr_A} = \frac{12}{3} = 4$ shots.

Prior to the first cycle A fires

$$k = \left[\frac{\delta r_A}{60} + 1\right] = \left[\frac{(10)(\frac{1}{2})}{60} + 1\right] = 1 \text{ shot.}$$

In the first cycle, B's 4 shots (his 1st, 2nd, 3rd, and 4th) are fired at times 10, 100, 190, 280 seconds; A's 3 shots (his 2nd, 3rd and 4th) are fired at times 120, 240 and 360 seconds. Thus the sequence of firers in each cycle is

BBABABA.

The transition matrices are

$$T_A = \begin{Vmatrix} .8 & .2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

and

$$T_B = \begin{Vmatrix} .9 & 0 & .1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

Thus

$$T = T_B\, T_B\, T_A\, T_B\, T_A\, T_B\, T_A = \begin{Vmatrix} .3359232 & .3626208 & .3014560 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

From equation 20

$$v_1 = .8$$

$$v_2 = .2$$

and from the above matrix,

$$t_1 = .3359232$$

$$t_2 = .3626208$$

$$t_3 = .3014560$$

$$t_4 = 0$$

The following table summarizes the results of this duel.

State	Description	Probability After m Cycles	Probability At End of Duel
1	Neither A nor B Wins	$(.8)(.3359232)^m$	0
2	A Wins	$.2 + .4368420\ (1-.3359232^m)$	.636842
3	B Wins	$.3631580\ (1-.3359232^m)$	.363158
4	A & B Kill Each Other	0	0

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